Solutions to Practice Problems for Exam 1

Aside from having problems of the type below, the exam with also have true-false type questions and short answer questions that test concepts and definitions. Of course, there are many more problems below than will be on the exam – since these are practice problems.

Please note:

(i) **No calculators will be allowed during the exam.**
(ii) You must take the exam in class at our usual time. Attendance will be taken.
(iii) The exam will be posted on *Canvas*, and this is where you will upload the copy of your solutions.

1. Put the following matrices in reduced row echelon form and identify the rank of each matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Solution. For $A$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \xrightarrow[1\cdot R_1+R_3]{-3\cdot R_1+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -5 \\ 0 & 4 & 6 \end{bmatrix} \xrightarrow{-\frac{1}{4}\cdot R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{5}{4} \\ 0 & 4 & 6 \end{bmatrix} \xrightarrow[-4\cdot R_2+R_3]{-1\cdot R_2+R_1} \begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-\frac{5}{4}\cdot R_3+R_2]{-\frac{3}{4}\cdot R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the rank of $A$ equals 3.

For $B$:

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix} \xrightarrow[-1\cdot R_1+R_3]{1\cdot R_1+R_2} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 1 \end{bmatrix} \xrightarrow{-1\cdot R_2+R_3} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}\cdot R_2} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Upon performing $-1 \cdot R_2 + R_1$ we have $\begin{bmatrix} 1 & 0 & -\frac{9}{5} & \frac{14}{5} \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank of $B$ is two.

For $C$:

$$\begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[1\cdot R_1+R_3]{-3\cdot R_1+R_2} \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow[\frac{1}{4}\cdot R_3]{1\cdot R_2+R_1} \begin{bmatrix} 1 & 0 & -5 & -12 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

If we now perform $8 \cdot R_3 + R_2$ and $5 \cdot R_3 + R_1$ we obtain $\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so that $C$ has rank 3.

2. Find the general solution (if it exists) for the system of equations corresponding to each of the following augmented matrices.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 3 & 1 \\ 0 & 1 & 6 & -1 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Solution. For $A$ the third and fourth variable are free variables. Using parameters $s, t$ we have

$$x_1 = 1 - 3s - 3t$$
$$x_2 = \sqrt{2} - 6s + t$$
$$x_3 = s$$
$$x_4 = t$$

For $B$: No solutions. For $C$: $x_1 = 3, x_2 = 3, x_3 = 4$.

3. Find the solutions (if they exist) to the systems of equations below. If the system is homogeneous, find a basic set of solutions:

$$\begin{aligned} 3x - 3y + 15z + w &= 0 \\ 4x - 4y + 20z &= 0 \\ 3w &= 0 \\ 2x - 2y + 10z &= 0 \end{aligned}$$

$$\begin{aligned} x + z &= 1 \\ y + z &= 2 \\ x + y &= 3 \end{aligned}$$

$$\begin{aligned} 2x + 3y - 7z &= 2 \\ x + y - z &= 4 \\ 6x + 8y - 16z &= 10 \end{aligned}$$

Solution. For the first system of equations:

$$\left[\begin{array}{cccc|c} 3 & -3 & 15 & 1 & 0 \\ 4 & -4 & 20 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 2 & -2 & 10 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & -4 & 20 & 0 & 0 \\ 3 & -3 & 15 & 1 & 0 \\ 2 & -2 & 10 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow[\frac{1}{2} \cdot R_3, \frac{1}{3} \cdot R_4]{\frac{1}{4} \cdot R_1} \left[\begin{array}{cccc|c} 1 & -1 & 5 & 0 & 0 \\ 3 & -3 & 15 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-1 \cdot R_1 + R_3]{-3 \cdot R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

After $-1 \cdot R_2 + R_4$ we have $\left[\begin{array}{cccc|c} 1 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. From this, we see that $x_1 = x_2 - 5x_3$, $x_4 = 0$. As . Using $s, t$, we have $x_1 = s - 5t, x_2 = s, x_3 = t, x_4 = 0$. As column vectors, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 5t \\ s \\ t \\ 0 \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

It follows that $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are basic solutions.

For the second system of equations:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right] \xrightarrow{-1 \cdot R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-1 \cdot R_3 + R_2]{-1 \cdot R_3 + R_1}$$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$. Therefore, $x = 1$, $y = 2$, $z = 0$.

For the third system of equations: Starting by interchanging the first two rows of the augments matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 2 & 3 & -7 & 2 \\ 6 & 8 & -16 & 10 \end{array}\right] \xrightarrow[-6 \cdot R_1 + R_3]{-2 \cdot R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -6 \\ 0 & 2 & -10 & -14 \end{array}\right] \xrightarrow{-2 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -6 \\ 0 & 0 & 0 & -2 \end{array}\right],$$

and thus, the system has no solution.

4. Convert the system of equations to a matrix equation, and then solve the system by finding the inverse to the coefficient matrix.

$$\begin{aligned} x + y + 2z &= 5 \\ x + y + z &= 0 \\ x + 2y + 4z &= -2 \end{aligned}$$

Solution. Setting $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$, the given system of equations becomes the matrix equation $A \cdot \mathbf{x} = \mathbf{b}$. Multiplying both sides of the matrix equation by $A^{-1}$ gives the solution $\mathbf{x} = A^{-1} \cdot \mathbf{b}$. To find $A^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-1 \cdot R_1 + R_3]{-1 \cdot R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[-1 \cdot R_3]{-1 \cdot R_2 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{-2 \cdot R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, and so

$$\mathbf{x} = A^{-1} \cdot \mathbf{b} = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -17 \\ 5 \end{bmatrix}.$$

Thus, $x = 12, y = -17, z = 5$.

5. For the matrices $A = \begin{bmatrix} 1 & 0 & -4 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 9 \\ 0 & 1 \\ 3 & 3 \end{bmatrix}$, Calculate $4A^2 + 5B^t \cdot C^t$.

Solution. We have

$$4 \cdot A^2 = 4 \cdot \begin{bmatrix} 1 & 0 & -4 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -4 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 & -8 & -20 \\ 8 & 11 & -1 \\ 4 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 4 & -32 & -80 \\ 32 & 44 & -4 \\ 16 & 64 & 72 \end{bmatrix}$$

and

$$5B^t \cdot C^t = 5 \cdot \begin{bmatrix} 2 & 0 \\ 2 & -1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 3 \\ 9 & 1 & 3 \end{bmatrix} = 5 \cdot \begin{bmatrix} 4 & 0 & 6 \\ -5 & -1 & 3 \\ 38 & 4 & 15 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 30 \\ -25 & -5 & 15 \\ 190 & 20 & 75 \end{bmatrix}.$$

Therefore,

$$4A^2 + 5B^t \cdot C^t = \begin{bmatrix} 4 & -32 & -80 \\ 32 & 44 & -4 \\ 16 & 64 & 72 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 30 \\ -25 & -5 & 15 \\ 190 & 20 & 75 \end{bmatrix} = \begin{bmatrix} 24 & -32 & 50 \\ 7 & 39 & 11 \\ 206 & 84 & 147 \end{bmatrix}.$$

6. For the matrix $A = \begin{bmatrix} 2 & -4 \\ 6 & 8 \end{bmatrix}$, find $A^{-1}$ in two ways: (i) using the formula involving the determinant of $A$ and (ii) using elementary row operations with an augmented matrix. Use your answer in (ii) to write $A^{-1}$ as a product of elementary matrices.

Solution. The determinant of $A$ equals 40. So $A^{-1} = \frac{1}{40} \cdot \begin{bmatrix} 8 & 4 \\ -6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{3}{20} & \frac{1}{20} \end{bmatrix}$. Using elementary row operations, we have:

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}\cdot R_2]{\frac{1}{2}\cdot R_1} \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ 3 & 4 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{-3\cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 10 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{\frac{1}{10}\cdot R_2} \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{20} & \frac{1}{20} \end{array}\right] \xrightarrow{2\cdot R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{1}{10} \\ 0 & 1 & -\frac{3}{20} & \frac{1}{20} \end{array}\right]$$

Rewriting the row operations as elementary matrices, it follows that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \cdot A = I_2,$$

and thus, $A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$.